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A FEW PROPERTIES OF CLIQUE GRAPHS

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ABSTRACT

We present solutions of several graph equations about the clique graphs, the line graphs, the middle graphs and the total graphs. Specially, the equation $C(L(G))=G$ generalizes the results of F. Escalante, S. T. Hedetniemi and P. J. Slater.

1. Introduction

We state the definitions and the notations required here. The definitions and the notations not presented here, may be found in Harary [4].

Let G be a simple graph. A clique of G is a maximal complete subgraph of G. Let $K(G)$ be the set of all cliques of G. The clique graph $C(G)$ of G is defined as having the elements of $K(G)$ as vertices and two vertices C_1, C_2 being adjacent in $C(G)$ if and only if the cliques C_1, C_2 have a nonempty intersection in G. Moreover, we define $C^n(G)$ by $C(C^{n-1}(G))$ $(n\geq 2)$.

For example, in Fig. 1, the subgraphs C_1, C_2 and C_3 are the cliques of G. The clique graph $C(G)$ and the graph $C^2(G)$ are also depicted in Fig.1.

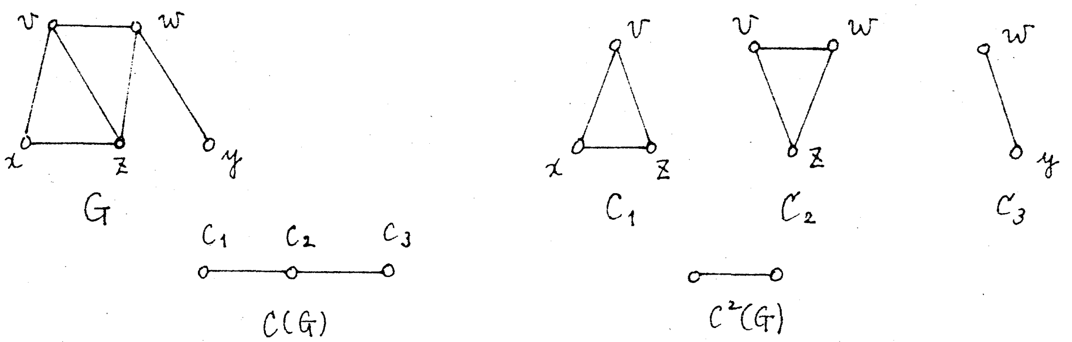


Fig.1

Next we define three special graphs of G. The line graph $L(G)$ of G is the graph with vertex set $E(G)$ and two vertices of $L(G)$ are adjacent if and only if they are adjacent edges of G. The middle graph $M(G)$ of G is the graph with vertex set $V(G) \cup E(G)$ and two vertices of $M(G)$ are adjacent if and only if 1) they are adjacent edges of G or 2) one is a vertex of G and another is an edge of G incident with it. The total graph $T(G)$ of G is the graph with vertex set $V(G) \cup E(G)$ and two vertices of $T(G)$ are adjacent if and only if 1) they are adjacent vertices or edges of G or 2) one is a vertex and another is an edge of G incident with it. Moreover we define $L^n(G)$, $M^n(G)$ and $T^n(G)$ by $L(L^{n-1}(G))$, $M(M^{n-1}(G))$ and $T(T^{n-1}(G))$ $(n \geqq 2)$ respectively.

For example, the line graph, the middle graph and the total graph of the graph G of Fig.1 are depicted in Fig.2.

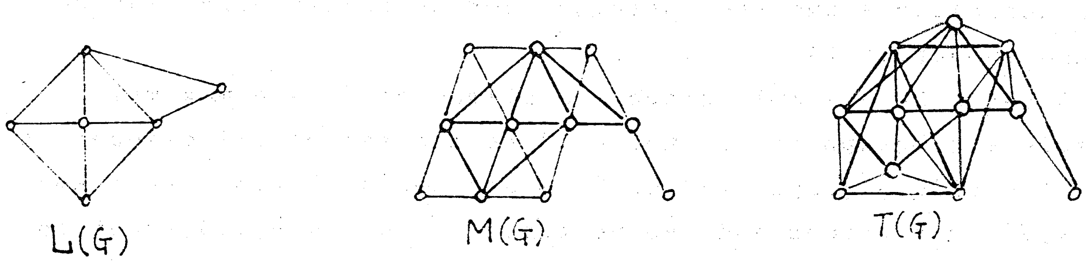


Fig.2

F. Escalante [2] gives the following result on the clique graphs having no triangles:

Proposition 1

A graph having no triangles satisfies the equation $C^2(G)=G$ if and only if every vertex of G is of degree at least two.

For example, in Fig.3, the graph G_1 satisfies the equation $C^2(G_1)=G_1$ but the graph G_2 does not. In fact it holds that $C^2(G_2)=G_2-\{u \; V(G_2) \mid deg \; u=1 \}$.

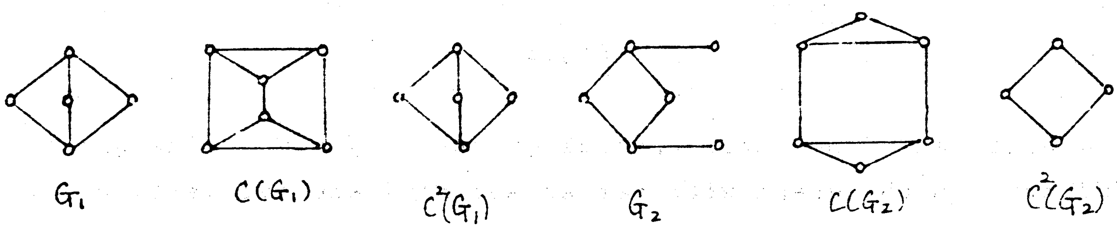


Fig.3

For this relation, S. T. Hedetnimi and P. J. Slater [5] obtain the following result :

Proposition 2

If G is a connected graph containing no triangles and at least three vertices, then it holds that $C^2(G)=C(L(G))=G-\{u\in V(G) \mid deg\ u=1\}$.

Thereupon, we can pose a problem which graphs satisfy the equation $C(L(G))=G$. We answer to this question in the next section. Moreover, by replacing $L(G)$ of the equation $C(L(G))=G$ by the middle graph $M(G)$ or by the total graph $T(G)$, we have the analogous equations and find the solutions to the equations: $C(M(G))=G$, $C(T(G))=G$.

2. The solutions of $C(L(G))=G$, $C(M(G))=G$ and $C(T(G))=G$.

Theorem 1

The graphs G satisfying the equation $C(L(G))=G$ are the only graphs which satisfies following three conditions:

(1) The degree of every vertex of G is at least two.

(2) Every pair of two triangles of G is edge-disjoint.

(3) Every triangle of G has exactly one vertex of degree two.

Proof. (<=) Suppose that G is a graph satisfying the three conditions. If G has not any triangle, then the equation $C(L(G))=G$ becomes $C^2(G)=G$, since in this case $C(G)\equiv L(G)$. Accordingly, $C(L(G))=G$ holds if and only if every vertex of G is of degree at least two by Proposition 1. We may consider in the case when G contains triangles. Construct the line graph $L(G)$ of G, then there corresponds to every triangle $T_j(j=1,\ldots,r)$ and to every vertex $v_k(k=1,\ldots,s)$ except for vertices u_j $(j=1,\ldots,r)$ of degree two on the triangle $L(T_j)=T_j'$ and $L(K_{1,\rho(v_k)})=K_{\rho(v_k)}$ $(\rho(v)=deg_G v)$, respectively. All T_j' and $K_{\rho(v_k)}$ are cliques of $L(G)$, and $L(G)$ has no cliques other than these. Now we define a mapping ϕ of $V(G)$ onto $V(C(L(G)))$ as follows:

$\phi(u_j)=T_j'(j=1,\ldots,r)$, if u_j is a vertex of degree two on T_j.

$\phi(v_k)=K_{\rho(v_k)}(k=1,...,s)$, if v_k is a vertex other than u_j $(j=1,...,r)$. $r+s=|V(G)|$.

The mapping ϕ has properties (α), (β) and (γ).

(α) The mapping ϕ is a bijection of $V(G)$ onto $V(C(L(G)))$.

(β) u_j and v_k are adjacent (v_k is a vertex of triangle T_j other than vertex u_j of degree two) in $G \iff T_j' \cap K_{\rho(v_k)} \neq \phi \iff \phi(u_j)$ and $\phi(v_k)$ are adjacent in $C(L(G))$.

(γ) Let both v_k, $v_j(k,j=1.,,,.s)$ be not vertices of degree two of triangle, then v_k, v_j are adjacent in $G \iff K_{\rho(v_k)} \cap K_{\rho(v_j)} \neq \phi$ in $L(G) \iff \phi(v_k)$ and $\phi(v_j)$ are adjacent in $C(L(G))$.

From (α), (β) and (γ), we know that ϕ is an isomorphism of G onto $C(L(G))$. Therefore, the graph G satisfying three conditions (1), (2) and (3) satisfies the equation $C(L(G))=G$.

($\Rightarrow$) Let G satisfy the equation $C(L(G))=G$ and condition (2) except for (1) or (3). If all vertices of a triangle of G have degree greater than two, then we have $C(L(G))=G \supset K_4$ (Fig.4). This contradicts to (2). Hence, at most two vertices of every triangle are of degree greater than two.

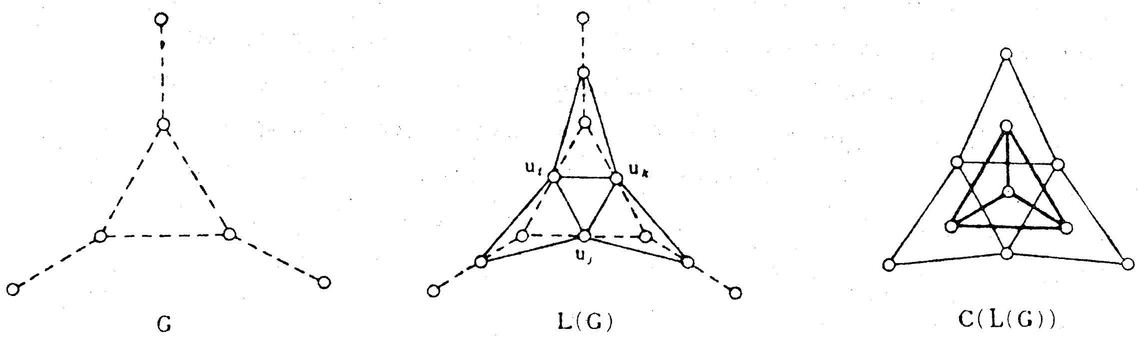


Fig.4

Then a clique K of $L(G)$ is one of the following cases:

(1) $K=L(K_{1,\rho(v)})=K_{\rho(v)}$, where either $v(v\in V(G))$ is on a triangle and $\rho(v)\geqq 3$, or v is on no triangle and $\rho(v)=2$.

(2) $K=L(T_j)$ $(j=1,...,r)$, where $\{T_1,...,T_r\}$ is a set of triangles of G.

Now we consider the following mapping ψ from $V(C(L(G)))$ to $V(G)$:

If $K=L(K_{1,\rho(v)})=K_{\rho(v)}$, then $\psi(K)=v$.

If $K=L(T_j)$ $(j=1,\ldots,r)$, then $\psi(K)=v$, where v is a vertex of degree two of a triangle T_j.
It is clear that ψ is an injection. Thus the number of cliques of $L(G)$ is at most $|V(G)|$. Consequently, if either G has a vertex of degree one, or two vertices of a triangle of G are of degree two, then we obtain an inequality $|V(C(L(G)))|>|V(G)|$. This contradicts the assumption $C(L(G))=G$.

Next, let G do not satisfy the condition (2) and contain an induced subgraph K_4-x which consists of two triangles having one vertex in common.

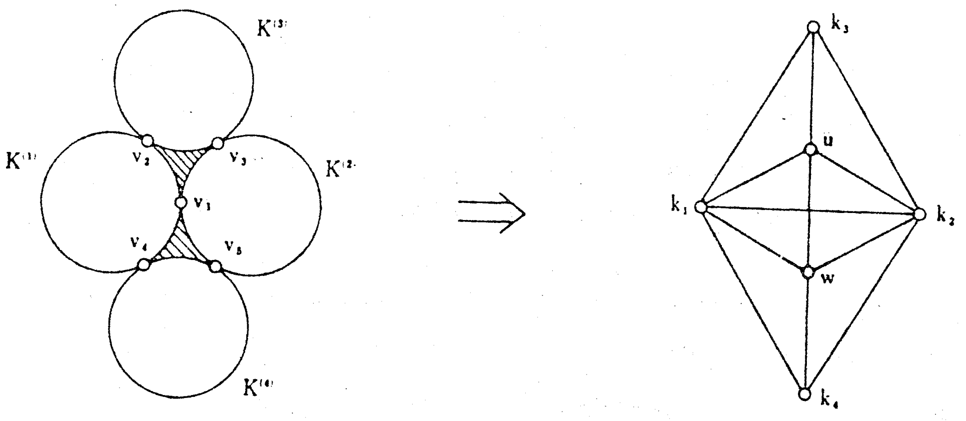
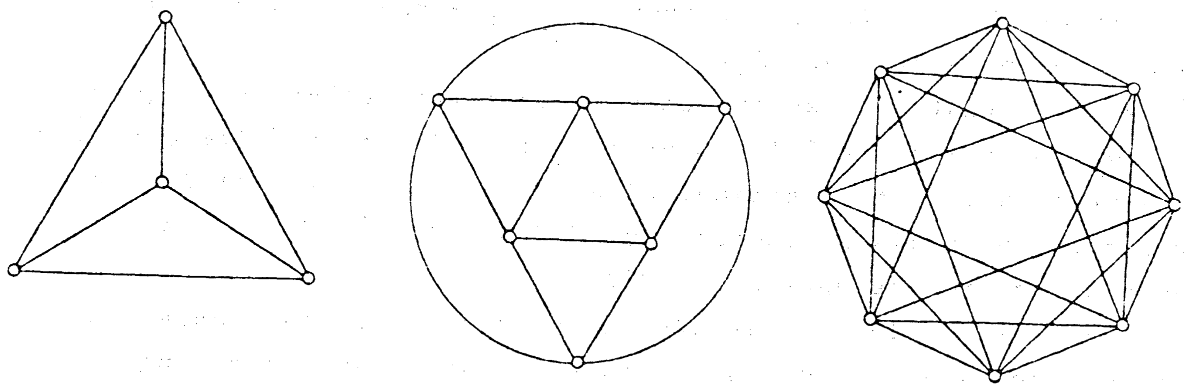


Fig.5

If G satisfies the equation $C(L(G))=G$, then there is a subgraph H of G such that in its line graph $L(H)$, complete subgraphs $K^{(1)}$, $K^{(2)}$ of $L(H)$ have a vertex in common with each of three complete subgraphs $K^{(j)}$ other than itself, respectively and complete subgraphs $K^{(3)}$, $K^{(4)}$ of $L(H)$ have a vertex in common with each of $K^{(1)}$, $K^{(2)}$, respectively (Fig.5). According to Krausz´s Theorem (Harary[4], Th.8.4), there may be such a graph.

But there occurs two triangles $<\{v_1,v_2,v_3\}>$, $<\{v_1,v_4,v_5\}>$ in $L(H)$ which have one vertex v_1 in common. Since these are complete subgraphs of $L(H)$, together with $K^{(i)}$ $(i=1,2,3,4)$, $C(L(H))$ contains a subgraph K_4, but is not isomorphic to K_4-x. Thus G must contain a subgraph K_4. Since $C(L(G))=G$ and G contains a subgraph K_4, G must contain $C(L(K_4))$ (Fig.6).

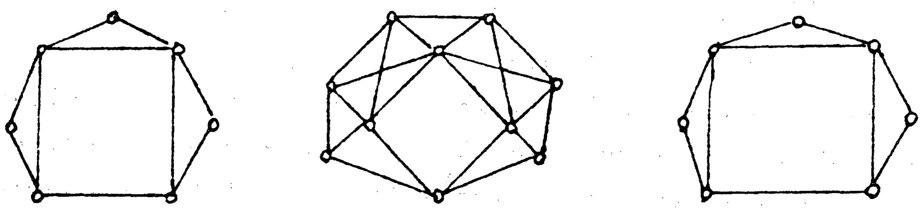


K_4 $L(K_4)$ $C(L(K_4))$

Fig.6

However, $C(L(K_4))$contains K_4, and so G contains $2K_4$. Similarly, $C(L(G))=G$ contains $2C(L(K_4))=4K_4$. Continuing this process, G becomes an infinite graph, but this is a contradiction. Hence, G does not satisfy the equation $C(L(G))=G$. //

For example, the graph H of Fig. 7 satisfies the conditions of Theorem 1 and the equation $C(L(H))=H$.



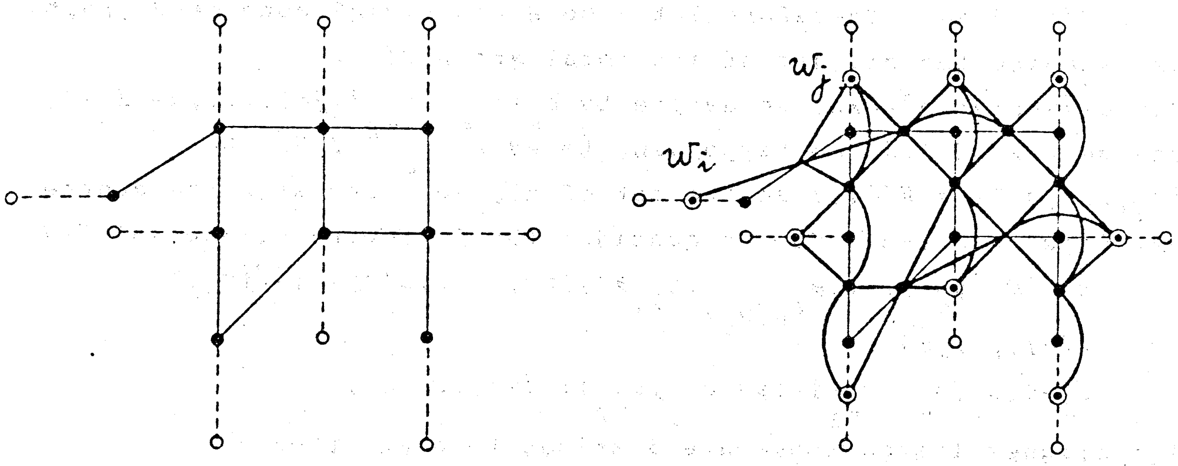
H $L(H)$ $C(L(H))$

Fig.7

Theorem 2

The graphs G satisfying the equation $C(M(G))=G$ are the only graphs which contain no triangles.

Proof. Let G^+ be the graph obtained by adding to G new p vertices v_i' $(i=1,\ldots,p)$ and new p edges $\{v_i, v_i'\}$, where $p=|V(G)|$ and $V(G)=\{v_1,\ldots,v_p\}$. Then $L(G^+)$ is isomorphic to $M(G)$ ([3],Th.1). Hence, the equation $C(M(G))=G$ considered may be rewritten as $C(L(G^+))=G$.



G^* (The broken lines are lines added to G.) $L(G^*)$(heavy lines)

(Fig.8)

(1) Let G contain no triangles. Drawing the line graph $L(G^+)$ of G^+, since G contains no triangles, in this case each clique of $L(G^+)$ is the line graph of the subgraph induced by some vertex v of G and its neighborhood in G^+, where the neighborhood of a vertex v of G is a set of all vertices being adjacent with v in G^+. Thus the number of cliques of $L(G^+)$ is equal to $|V(G)|$.

Each vertex of $L(G^+)$ belongs to at most two cliques ([4],Th.8.4). The vertex is unicliqual if it is in exactly one clique (as vertices w_i, w_j etc, in Fig.7). According to [4],Th.8.3, G^+ is obtained by acting L^{-1} to $L(G^+)$. In this case, by neglecting unicliqual vertices, L^{-1} becomes the operation C, and then G is obtained.

(2) Suppose that G contains triangles. All three vertices v_i, v_j, v_k of a triangle $<\{v_i,v_j,v_k\}>$ of G have degree of at least three in G^+. Therefore $L(G^+)$ has more cliques than G by at least one clique. Hence we have the inequality $|V(G)|<|V(C(L(G)))|$ i.e. $C(M(G))=C(L(G^+))\neq G$. //

Theorem 3

The graphs G satifying the equation $C(T(G))=G$ are only totally disconnected graphs.

Proof. If G is a totally disconnected graph, then it is clear that $C(T(G))=G$. Therefore let G be a nontrivial connected graph, and examine the cliques of its total graph $T(G)$.

For convenience sake, we denote by $e_j=\{v_{j_1},v_{j_2}\}(j=1,\ldots,n=|E(G)|)$ the vertex of $T(G)$ corresponding to edge e_j of G. Let $\{v_1,\ldots,v_m\}(m=|V(G)|)$ be the set of all vertices with the degree not less than two. Now we consider the following cliques of $T(G)$:

$$K^{(i)}=\langle\{e_{j_1},\ldots,e_{j_{\rho(v_i)}}\mid e_{j_t}\ni v_i(t=1,\ldots,\rho(v_i))\}\cup\{v_i\}\rangle$$
$(i=1,\ldots,m)$
$$L_j=\langle\{e_j,v_{j_1},v_{j_2}\}\rangle(e=\{v_{j_1},v_{j_2}\};\ j=1,\ldots,n).$$

All cliques listed above are distinct to each other.

Next, for each vertex v_i $(i=1,\ldots,|V(G)|)$ of G, correspond to a clique of $T(G)$:

If $\rho(v_i)=1$, then v_i corresponds to L_j, where $e_j=\{v_i,v_k\}$. $(\#)$

If $\rho(v_i)\geqq 2$, then v_i corresponds to $K^{(i)}$ $(i=1,\ldots,m)$.

It is clear that the total number of cliques in $(\#)$ is equal to $|V(G)|$. Let G contain a triangle $T=\langle\{v_i,v_j,v_k\}\rangle$. Then a subgraph $M=\langle\{\{v_i,v_j\},\{v_j,v_k\},\{v_k,v_i\}\}\rangle$ of $T(G)$ is a clique of $T(G)$ and different from every cliue in $(\#)$. That is, we have $|V(G)|<|V(C(T(G)))|$. Hence G must be a star $K_{1,s}$ $(s\geqq 1)$. But, if s is not less than two, then we have $C(T(K_{1,s}))=K_{1+s}$; if s is equal to one, then we have $C(T(K_{1,1}))=K_1$. Thus it is clear that $C(T(G))\neq G$. From the above, it follows that there does not exist any non-trivial connected graph which satisfies given equation. //

2. The solutions of $C(L^n(G))=G$, $C(M^n(G))=G$ and $C(T^n(G))=G$ $(n\geqq 2)$.

In this section, we replace L with $L^n(n\geqq 2)$ etc, and present the solutions of the equations $C(L^n(G))=G$, $C(M^n(G))=G$ and $C(T^n(G))=G$ $(n\geqq 2)$.

At first we consider the solutions of the equation $C(L^n(G))=G$ $(n\geqq 2)$.

<u>Lemma 4.1.</u>

If a graph G has a triangle as a subgraph, then the graph $C(L^n(G))$ is not isomorphic to G $(n\geqq 2)$.

<u>Proof.</u> Suppose that there is a graph G with triangles satisfying the equation $C(L^n(G))=G$ $(n\geqq 2)$. Let G be connected.

If G is a triangle, then it holds that $C(L^n(G))=K_1 (n\geqq 1)$ i.e. $C(L^n(G))\neq G(n\geqq 2)$. Suppose that G has a triangle as a properly subgraph. Then there is a vertex v being adjacent to one vertex of this triangle. That is, G has the graph G_1 of Fig.9 as a subgraph. In the case of $n=2$, G has the graph K_4 as a subgraph since $C(L^2(G_1))=K_4$ (see Fig.9).

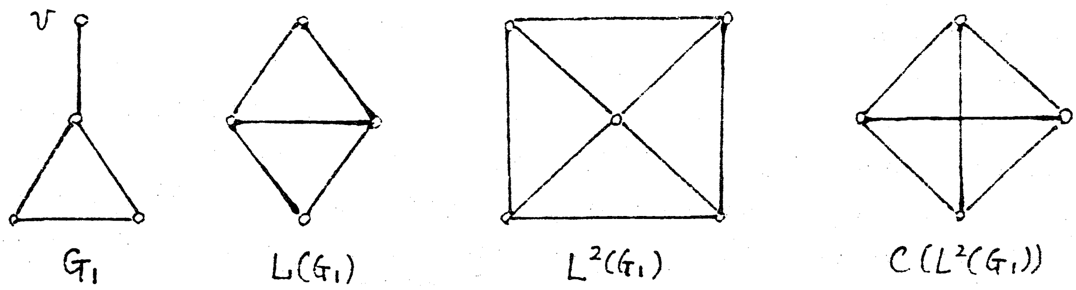


Fig.9

In the case of $n=3$, G has the graph K_4 since the graph $C(L^3(G_1))$ has the graph K_4 (see Fig.10).

Fig.10

The line graph $L^3(G_1)$ has the graph T (see Fig.11). Then it holds that the graph $C(L^4(G_1))$ has the graph $C(L(T))$. In the case of $n=4$, G has the graph K_4 since the graph $C(L^4(G_1))$ has the graph K_4.

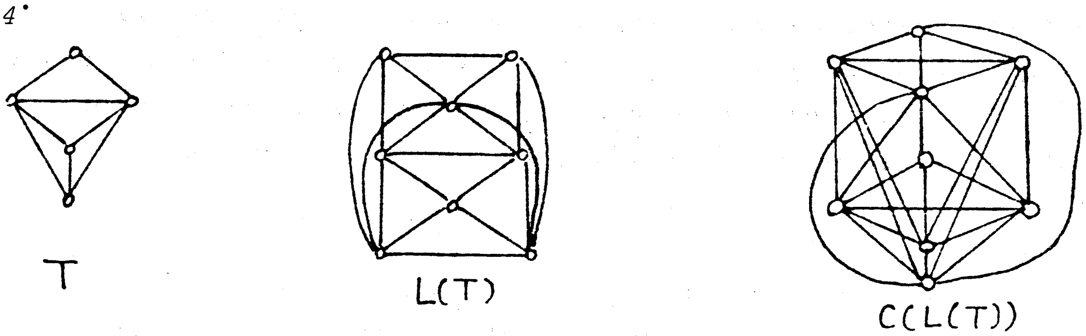


Fig.11

In the case of $n=5$, since $L^4(G_1)\geqq L(T)\geqq T$, the graph $C(L^5(G_1))$ has the graph $C(L(T))$, i.e. K_4 as a subgraph. That is, G has the graph K_4. Similarly to the case of $n=5$, in the case of $n\geqq 6$, G has the graph K_4 as a subgraph. Hence G has K_4 for any case of $n\geqq 2$.

Next we shall show that G is an infinite graph and obtain a contradiction.

Case 1 $n=2$.

The graph $C(L^2(K_4))$ has the graph $2K_4$ (see Fig.12). Since G has the graph $2K_4$, $G=C(L^2(G))$ has the graph $C(L^2(2K_4))$, i.e. $4K_4$. In general, G has the graph 2^nK_4 for any $n\geqq 1$. Hence G is an infinite graph. This is a contradiction to that G is finite.

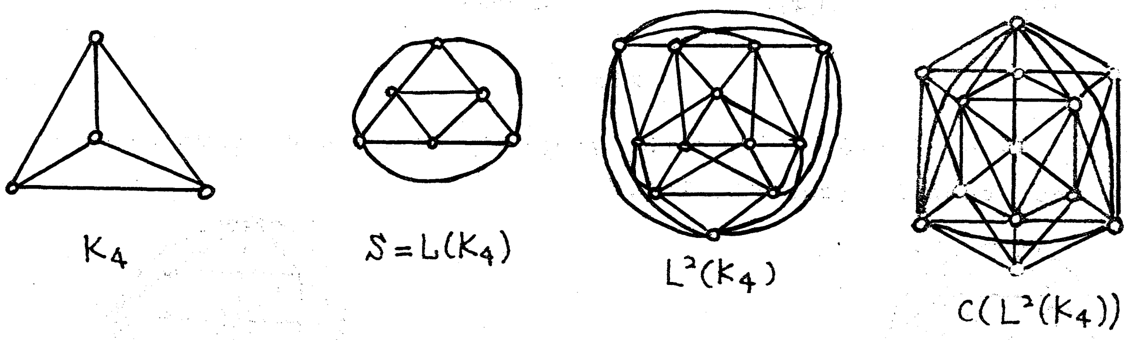


Fig.12

Case 2 $n\geqq 3$.

Set $S=L(K_4)$. Then $L(S)$ has K_4 (see Fig.12), so that the line graph $L^3(K_4)=L^2(S)$ has the line graph $L(K_4)=S$. That is, the graph $C(L^3(K_4))$ has the graph $C(S)$ i.e. $2K_4$ (see Fig.6). Since the graph $L^4(K_4)$ has the graph $L(S)$, the graph $C(L^4(K_4))$ has the graph $C(L(S))$, i.e. $2K_4$ (see Figl2). Moreover, since $L^5(K_4)\geqq L^2(S)\geqq L(K_4)$ $=S$, the graph $C(L^5(K_4))$ has the graph $C(S)$ i.e. $2K_4$ (see Fig.6). Similarly to the case of $n=5$, in the case of $n\geqq 6$, the graph $C(L^n(K_4))$ has the graph $2K_4$ as a subgraph. Since G has the graph $C(L^n(K_4))$ as a subgraph, G has $2K_4$ for any $n\geqq 3$. Hence G has the graph 2^nK_4 for any n. Therefore G is an infinite graph, this being a contradiction. //

We must note that if H is a subgraph of a graph G, then the graph $C(L^n(H))$ is a subgraph of the graph $C(L^n(G))$ $(n\geqq 2)$.

Lemma 4.2

Let n be two or more. Then, if a graph G has no triangles and there is a vertex of G such that $deg\ v \geqq 4$, then the graph $C(L^n(G))$ is not isomorphic to G.

Proof. Let G be a graph which satisfies the condition of Lemma and the equation $C(L^n(G))=G$ $(n \geqq 2)$. Then G has the star $K_{1,4}$ as an induced subgraph. In the case of $n=2$, the graph $C(L^2(K_{1,4}))=C(L(K_4))$ has the complete graph K_4 as a subgraph. That is, G has K_4. This is a contradiction. In the case of $n \geqq 3$, since $n-1 \geqq 1$, the graph $C(L^n(K_{1,4}))=C(L^{n-1}(K_4))$ has the complete graph K_4 as a subgraph by the similar argument to the proof of Lemma 4.1. But this is a contradiction. //

Lemma 4.3

Let n be two or more. Then, if a graph G has no triangles and there is a vertex of G such that $deg\ w=3$, and the degree of each vertex of G is two or three, then the graph $C(L^n(G))$ is not isomorphic to G.

Proof. Let G be a connected graph which satisfies the condition of Lemma and the equation $C(L^n(G))=G$ $(n \geqq 2)$. Then G contains one of the graphs in Fig.13 as a subgraph.

Fig.13

Case 1. G has the graph H_1.

Then the graph $C(L^2(H_1))$ is a triangle. Hence G has a triangle. Similarly to the proof of Lemma 4.1, it follows that G has a triangle for any $n \geqq 2$. This is a contradiction.

Case 2. G has the graph H_2.

Similarly to Case 1, we obtain a contradiction. //

Lemma 4.4

Let n be two or more. Then, if every vertex of a graph G having no triangles has the degree three or less and some vertex of G has the degree one, then the graph $C(L^n(G))$ is not isomorphic to G.

Proof. Let G be a connected graph which satisfies the condition of Lemma and the equation $C(L^n(G))=G$ $(n \geqq 2)$. Then G is one of the following graphs:

1) a path.
2) a graph containing a cycle of length four or more as a properly subgraph.
3) a tree distinct to a path.

1) $G=P_m (m \geqq 1)$.

It holds that

$$C(L^n(P_m))=\begin{cases} P_{m-n-1} & (m>n+1) \\ K_1 & (m=n+1) \\ \Phi & (m<n+1) \end{cases}$$

where Φ denotes the empty graph. Hence $C(L^n(P_m))$ is not isomorphic to P_m.

2) G contains a cycle $C_m (m \geqq 4)$ as a properly subgraph.

G has one of the graphs in Fig.14.

Fig.14

Similarly to the proof of Lemma 4.1, it follows that G has a triangle for any $n \geqq 2$. This is a contradiction.

3) G is a tree but not path.

Then G has the star $K_{1,3}$. If G is the star $K_{1,3}$, then we have $C(L^n(K_{1,3}))=K_1 \neq K_{1,3} (n \geqq 2)$. Accordingly, we suppose that G has $K_{1,3}$ as a properly subgraph.

Case 1. $n \geqq 3$.

G has the graph T_1 in Fig. 15.

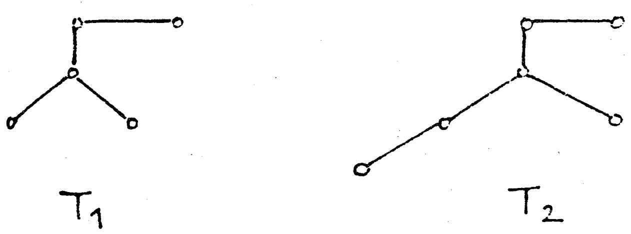
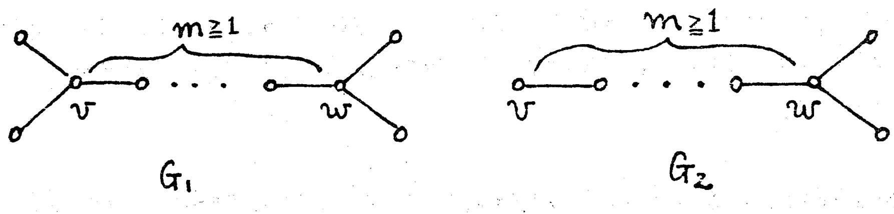


Fig.15

Similarly to the proof of Lemma 4.1, it follows that G has a triangle.

Case 2. $n=2$.

Since the graph $C(L^2(T_2))$ is a triangle, cannot contain T_2. Therefore G is isomorphic to the graphs in Fig.16.



(m is the length of vw-path.)

Fig.16

But, since $C(L^2(G_1))=P_{m+2}$ and $C(L^2(G_2))=P_m$, $C(L^2(G))$ is not isomorphic to G. //

By these lemma and noting that $C(L^n(C_m))=C_m (m \geqq 4)$, we obtain the following theorem.

<u>Theorem 4.</u>

The graphs G satisfying the equation $C(L^n(G))=G (n \geqq 2)$ are only regular graphs of degree two not including triangles.

Moreover we have the following theorem.

<u>Theorem 5.</u>

The graphs G satisfying the equation $C(M^n(G))=G (n \geqq 2)$ are only totally disconnected graphs.

Proof. Similar to the proof of theorem 4. //

Theorem 6.

The graphs G satisfying the equation $C(T^n(G))=G\,(n\geqq 2)$ are only totally disconnected graphs.

Proof. Let G be a connected (p,q)-graph. Since $C(T^n(K_1))=K_1$, we suppose that $G\neq K_1$. Moreover we set

$$V(G)=\{v_1,\ldots,v_r,w_1,\ldots,w_{p-r}\}\quad \deg v_i=1\,(i=1,\ldots,r),\ \deg w_j\geqq 2\,(j=1,\ldots,p-r),$$

$$E(G)=\{e_1,\ldots,e_q\}.$$

Now we define the mapping $\phi : V(G)\to V(C(T(G)))$ as follows:

$$\phi(v_i)=\langle\{v_i,e_{j_i},v_{i_1}\}\rangle_{T(G)}\quad e_{j_i}=\{v_i,v_{i_1}\}\,(i=1,\ldots,r),$$

$$\phi(w_j)=\langle\{w_j,w_{j_1},\ldots,w_{j_{\rho(w_j)}}\}\rangle_{T(G)}\quad N(w_j)=\{w_{j_1},\ldots,w_{j_{\rho(w_j)}}\}\quad (j=1,\ldots,p-r).$$

Then ϕ is an injection from $V(G)$ into $V(C(T(G)))$. Hence it holds that $|V(G)|\leqq|V(C(T(G)))|$. Repeating this process, we have $|V(G)|=p\lneqq p+q=|V(T(G))|\leqq|V(C(T^2(G)))|$. Generally, it holds that $|V(G)|\lneqq|V(C(T^n(G)))|$ for any $n\geqq 2$. Hence $C(T^n(G))$ is not isomorphic to G. //

4. The solutions of $C^2(L^m(G))=G$, $C^2(M^m(G))=G$ and $C^2(T^n(G))=G$ $(m\geqq 2,\ n\geqq 1)$.

At first, we prove the proposition required to reseach the solutions of $C^2(L^n(G))=G$ and $C^2(M^n(G))=G$ $(n\geqq 2)$.

Proposition 3.

Let G be a connected simple graph and v a vertex of G. Then $C^2(L(G-v))$ is a subgraph of $C^2(L(G))$.

Proof. Since $L(G-v)$ is an induced subgraph of $L(G)$, $C(L(G-v))$ is a subgraph of $C(L(G))$ by Escalante[2]. Hence there is a unique clique $K_i\in K(L(G))$ corresponding to each clique $K_i'\in K(L(G-v))$ $(i=1,2,\ldots,m=|K(L(G-v))|)$.

Now we shall that $K_i'\cap K_j'=\phi\,(i\neq j;\,i,j=1,\ldots,m)\Rightarrow K_i\cap K_j=\phi$. If so, then $C(L(G-v))$ is an induced subgraph of $C(L(G))$. Therefore $C^2(L(G-v))$ is a subgraph of $C^2(L(G))$ by Escalante[2].

We divide three cases.

Case 1. Both K_i' and K_j' are the line graphs of a triangle in $G-v$.

Then, since $K_i=K_i'$ and $K_j=K_j'$, it holds that $K_i \cap K_j=\phi$.

Case 2. K_i' is the line graph of a triangle T in $G-v$ and K_j' is the line graph of a subgraph induced by the closed neighborhood of a vertex w in $G-v$.

Since $d_{G-v}(V(T),w) \geqq 1$, we have $d_G(V(T),w) \geqq 1$, where $d_G(V(T),w) = \min_{v \in V(T)} d_G(v,w)$ etc. Hence we have $K_i \cap K_j=\phi$.

Case 3. Both K_i' and K_j' are the line graphs of a subgraph induced by the closed neighborhood of a vertex in $G-v$.

K_i', K_j' are constructed by vertices w,z in $G-v$ respectively. Since $d_{G-v}(w,z) \geqq 2$, we have $d_G(w,z) \geqq 2$, i.e. $K_i \cap K_j=\phi$. //

Cor.1

Let G be a simple graph and H a subgraph of G. Then the graph $C^2(L^n(H))$ is a subgraph of the graph $C^2(L^n(G))(n \geqq 2)$.

Cor.2

Let G be a simple graph and H a subgraph of G. Then the $C^2(M^n(H))$ is a subgraph of the graph $C^2(M^n(G))(n \geqq 2)$.

Proof. By $M(G)=L(G^+)$ (Hamada and Yoshimura [3]) and Cor.1. //

By Cor.1 and Cor.2, we can perform the same argument as the proof of Lemma 4.1. Before we state theorem 7, we prove a few lemmas.

Lemma 7.1

Let G be a graph containing no triangles and satisfying the condition $2 \leqq \delta(G) \leqq \Delta(G) \leqq 3$. Then G is isomorphic to $C^2(L^2(G))$.

Proof. We put the vertex set $V(G)$ and the edge set $E(G)$ of G as follows:

$V(G)=\{v_1,\ldots,v_r,w_1,\ldots,w_{p-r}\}$,

$deg_G v_i=3$, $deg_G w_j=2 (i=1,\ldots,r; j=1,\ldots,p-r; p=|V(G)|)$,

$E(G)=\{e_1,\ldots,e_q\}$ $(q=|E(G)|)$.

Then we can classify the cliques of $L^2(G)$ to the following two types:

$K_i=L(<\{e_{i_1},e_{i_2},e_{i_3}\}>_{L(G)})$ ($v_i \in V(G)$, $e_{j_i} \ni v_i$, $i=1,\ldots,r$; $=1,2,3$),

$M_k=L(<\{e_k e \in E(L(G)) \mid e_k$ and e are adjacent in $L(G)\}>)$, ($e_k \in E(G); k=1,\ldots,q$).

Therefore we can classify the cliques of $C(L^2(G))$ to the following two types:

$\bar{K}_i=<\{M_{i_1},M_{i_2},M_{i_3},K_i\}>_{C(L^2(G))}$, where $e_{i_j} \ni v_i$ ($i=1,\ldots,r$; $j=1,2,3$) and $M_{i_j},K_i \in K(L^2(G))$.

$\bar{M}_j=<\{M_{j_1},M_{j_2}\}>_{C(L^2(G))}$, where $e_{j_i} \ni w_i$ ($j=1,\ldots,p-r; i=1,2$) and $M_{j_1},M_{j_2} \in K(L^2(G))$.

Here any two cliques of these cliques are distinct and the set of these cliques coincides with the set of all cliques of $C(L^2(G))$.

Now we construct the mapping $\phi : V(G) \rightarrow V(C^2(L^2(G)))$ as follows:

$\phi(v_i)=\bar{K}_i$ ($i=1,\ldots,r$),

$\phi(w_j)=\bar{M}_j$ ($j=1,\ldots,p-r$).

Then ϕ is a bijection. Thereupon we shall show that ϕ preserves the adjacency between the vertices of $C^2(L^2(G))$.

(1) v_i and w_j are adjacent in $G(i=1,\ldots,r; j=1,\ldots,p-r)$,

<=> $K_i \cap M_k \ni \{\{e_k,e_h\}\}$ in $L^2(G)$, where $K_i=L(<\{e_k,e_h,e_l\}>)$,

<=> $\bar{K}_i \cap \bar{M}_k \ni M_k$,

<=> $\phi(v_i)$ and $\phi(w_j)$ are adjacent in $C^2(L^2(G))$.

(2) v_{i_1} and v_{i_2} are adjacent in $G(i_1,i_2=1,\ldots,r)$,

<=> $\bar{K}_{i_1} \cap \bar{K}_{i_2} \ni M_i$ where $e_i=\{v_{i_1},v_{i_2}\} \in E(G)$,

<=> $\phi(v_{i_1})$ and $\phi(v_{i_2})$are adjacent in $C^2(L^2(G))$.

(3) w_{j_1} and w_{j_2} are adjacent in $G(j_1,j_2=1,\ldots,p-r)$,

<=> $\bar{M}_{j_1} \cap \bar{M}_{j_2} \ni M_j$ where $e_j=\{w_{j_1},w_{j_2}\} \in E(G)$,

<=> $\phi(w_{j_1})$ and $\phi(w_{j_2})$ are adjacent in $C^2(L^2(G))$.

Here ϕ is an isomorphism between G and $C^2(L^2(G))$. //

<u>Lemma 7.2.</u>

Let G be a graph containing no triangles and satisfying the condition $\delta(G)=1$ and $\Delta(G)\leqq 3$. Then G is not isomorphic to $C^2(L^2(G))$.

Proof. Similarly to the proof of Lemma 7.1, we obtain the result that $|V(C^2(L^2(G)))|<|V(G)|$. Hence we have $C^2(L^2(G))\neq G$. //

Theorem 7.

Let G be a connected graph. Then G satisfies the equation $C^2(L^n(G))=G(n\geqq 2)$ if and only if $n=2$ and G has no triangles and satisfies the condition $2\leqq\delta(G)\leqq\Delta(G)\leqq 3$.

Proof. By Lemma 7.1,7.2 and the similar argument to the proof of theorem 4. //

For example, the graph G of Fig.17 satisfies the condition of theorem 7 and the equation $C^2(L^2(G))=G$.

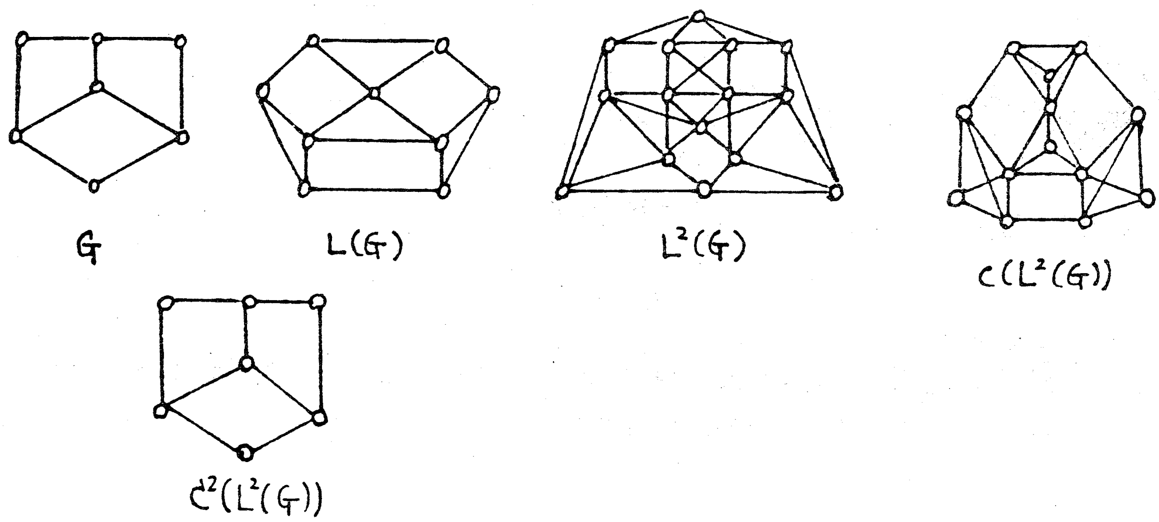


Fig.17

Theorem 8.

Let G be a connected graph. Then G satisfies the equation $C^2(M^n(G))=G(n\geqq 2)$ if and only if $n=2$ and G is a path or a cycle.

Proof. Similar to the proof of theorem 7. //

For example, Fig.18 shows that $C^2(M^2(P_4))=P_4$ and $C^2(M^2(C_4))=C_4$.

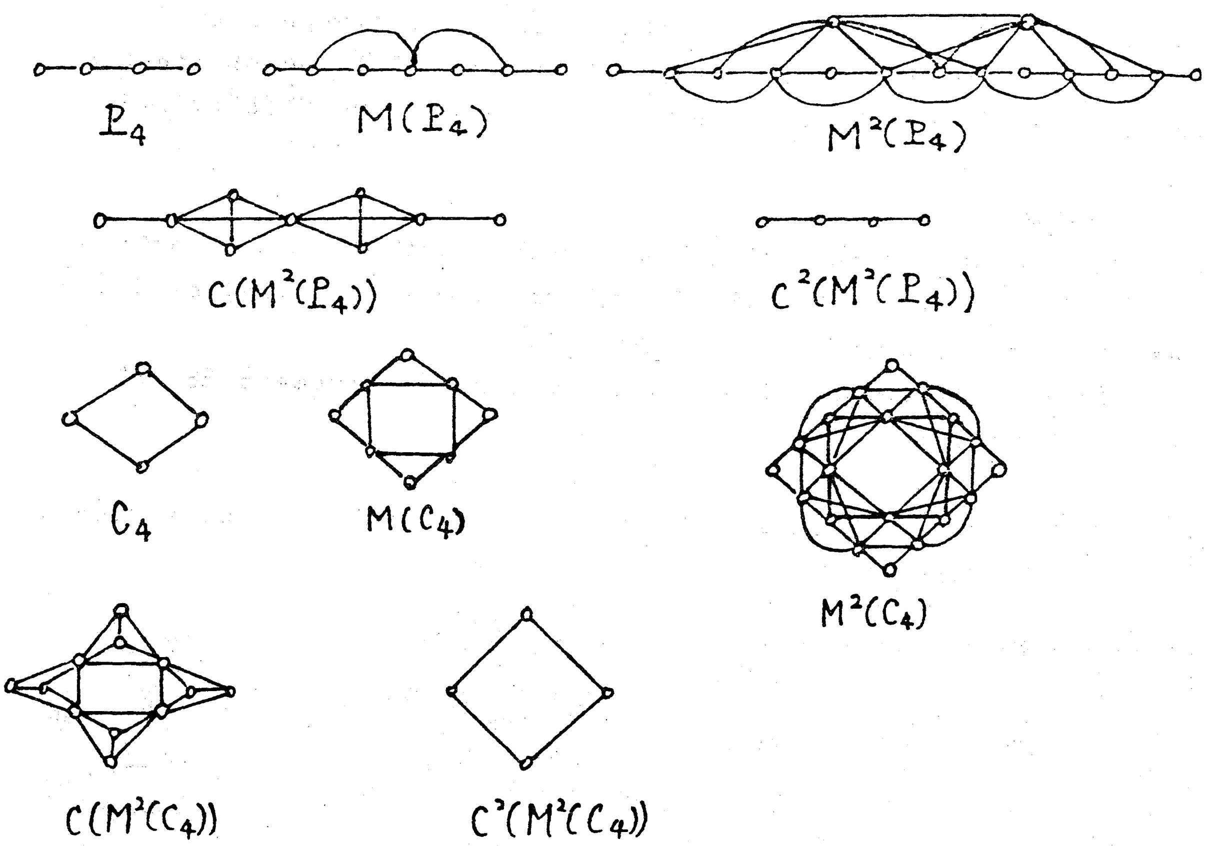


Fig.18

Theorem 9.

The graphs G satisfying the equation $C^2(T^n(G))=G(n\geqq 1)$ are only totally disconnected graphs.

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